ON THE COMPLEXITY OF TOPOLOGICAL CONJUGACY OF TOEPLITZ SUBSHIFTS

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Abstract. In this paper, we study the descriptive set theoretic complexity of the equivalence relation of conjugacy of Toeplitz subshifts of a residually finite group G. On the one hand, we show that if $G = \mathbb{Z}$, then topological conjugacy on Toeplitz subshifts with separated holes is amenable. In contrast, if G is non-amenable, then conjugacy of Toeplitz G-subshifts is a non-amenable equivalence relation. The results were motivated by a general question, asked by Gao, Jackson and Seward, about the complexity of conjugacy for minimal, free subshifts of countable groups.

1. Introduction

The theory of definable equivalence relations offers tools for classifying the complexity of equivalence relations arising from various isomorphism problems. An important class of Borel equivalence relations is given by the *countable* ones, i.e., those with countable classes. It is a classical result of Feldman and Moore that every such equivalence relation arises as the orbit equivalence relation of a Borel action of a countable group; and thus, from the beginning, the theory is intimately connected to that of dynamical systems from where it has borrowed most of its tools and techniques. The descriptive set theoretic approach to those equivalence relations has been developed over the last twenty years by Dougherty, Jackson, Kechris, Louveau, Hjorth, Thomas and others (see, e.g., [13, 5, 12]).

The natural comparison of equivalence relation is given by *Borel reducibility* and the simplest ones are those which are *smooth*, i.e., admit real numbers as complete invariants (or, equivalently, admit a Borel transversal). The next level of the complexity hierarchy is formed by the *hyperfinite* ones, equivalently, those induced by Borel actions of the group of integers $\mathbb{Z}$. There is also a *universal* countable Borel equivalence relation, which is maximal in the quasi-order of Borel reducibility, an example is the orbit equivalence relation $F_2 \curvearrowright 2^{F_2}$.

In topological dynamics, one of the most commonly considered types of dynamical systems are the *subshifts* (also known as *Bernoulli subflows* or *symbolic dynamical systems*). For a countable group G, the *Bernoulli shift* is the action of G on 2^G defined by the formula: $(g \cdot x)(h) = x(g^{-1}h)$ for all $g, h \in G$, $x \in 2^G$. A G-*subshift* is a closed nonempty subset $S \subseteq 2^G$ which is invariant under the action of G. The natural isomorphism relation of subshifts is *topological conjugacy*: two subshifts S and T are *conjugate* if there is a homeomorphism $f : S \to T$ which commutes with the action of G. This is the equivalence relation that we study in this paper. Often, special classes of subshifts are of interest in dynamics. A subshift S is *minimal* if it does not contain any proper subshift, or equivalently, if every orbit is dense. It is *free* if for any $x \in S$ and any $g \in G$ different from 1_G, we have $g \cdot x \neq x$.

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It is worth noting that while many countable equivalence relations arise naturally from group actions, some do not, and their study is usually more difficult because most of the available tools are dynamical in nature and require the presence of a (natural) group action. Some notable examples where there is no natural group action that gives the equivalence relation are Turing equivalence, isomorphism of (various classes of) finitely generated groups, and topological conjugacy of subshifts. (However, for subshifts, there is still a group action present that can be exploited: see Section 4.)

The recent monograph of Gao, Jackson, and Seward [9] studies the complexity of topological conjugacy of free, minimal subshifts. (In fact, before their construction, it was an open problem whether such subshifts necessarily exist for every countable G.) It follows essentially from a classical result of Curtis, Hedlund and Lyndon (see [19] or [9, Lemma 9.2.1]) that for any countable group G, topological conjugacy of G-subshifts is a countable Borel equivalence relation. Gao, Jackson and Seward showed [9, Corollary 1.5.4] that if G is infinite, this equivalence relation is not smooth, and that if G is locally finite, then it is hyperfinite [9, Theorem 1.5.6]. They also pose the general question [9, Problem 9.4.11] to determine the complexity of this equivalence relation for an arbitrary countable group G. This was an important motivating question for our work and in Theorem 1.2, we provide a partial answer.

Clemens [1] proved that the topological conjugacy of $\mathbb{Z}$-subshifts is a universal countable Borel equivalence relation. However, his construction produces subshifts that are far from minimal and it remains an open question whether isomorphism of minimal $\mathbb{Z}$-subshifts is universal. This question is connected with the conjecture of Thomas [23, Conjecture 1.2] that isomorphism of finitely generated, amenable, simple groups is universal. It follows from the results of Matui [21], Giordano, Putnam, Skau [10] and a recent result of Juschenko and Monod [14] that the computation of the topological full group of a minimal $\mathbb{Z}$-subshift provides a reduction from the (flip-) conjugacy of minimal $\mathbb{Z}$-subshifts to isomorphism of finitely generated, simple, amenable groups. Another related result was recently proved by Williams [24], who showed that isomorphism of finitely generated, solvable groups is weakly universal.

The focus of this paper is studying the complexity of the conjugacy equivalence relation for the class of *Toeplitz subshifts* of residually finite groups. This class of subshifts is well-known and appears in many contexts. For example, Downarowicz [7] showed that that any Choquet simplex can be realized as the simplex of invariant measures of a Toeplitz subshift. (This result was recently generalized to arbitrary amenable, residually finite groups by Cortez and Petite [4].) An important feature of Toeplitz words is that they can be constructed in stages, which allows a fair amount of control. We briefly recall the classical definition for $G = \mathbb{Z}$ here and postpone the general one for residually finite groups to Section 2.

A word $x \in 2^{\mathbb{Z}}$ is *Toeplitz* if every symbol occurs periodically, i.e., for every $n \in \mathbb{N}$, there exists k such that $x(n + ki) = x(n)$ for all $i \in \mathbb{Z}$. A subshift $S \subseteq 2^{\mathbb{Z}}$ is *Toeplitz* if it is equal to the closure of the orbit of some Toeplitz word. It is easy to check that every Toeplitz subshift is minimal.

The topological conjugacy relation for Toeplitz $\mathbb{Z}$-subshifts has been studied by Downarowicz, Kwiatkowski and Lacroix [6] and it essentially follows from their results that topological conjugacy of *pointed* Toeplitz flows (i.e., the relation E on Toeplitz words, such that $x \mathrel{E} y$ if there is a an $\mathbb{Z}$-equivariant homeomorphism from $\overline{\mathbb{Z} \cdot x}$ to $\overline{\mathbb{Z} \cdot y}$ that maps x to y) is a hyperfinite equivalence relation. The latter seems to indicate that the topological conjugacy relation for Toeplitz $\mathbb{Z}$-subshifts should not be too complicated. However, we have only been able to treat the case

of Toeplitz subshifts *with separated holes*, a special but important class. The first of our main results is the following.

Theorem 1.1. *For $G = \mathbb{Z}$, the equivalence relation of conjugacy on Toeplitz subshifts* with separated holes *is amenable and therefore hyperfinite μ-a.e. for every Borel probability measure μ on the set of subshifts.*

We postpone the definitions to Section 3 and just recall that an amenable equivalence relation is hyperfinite a.e. with respect to any probability measure but it is an open problem whether it must be hyperfinite everywhere. We do not know whether the above equivalence relation is hyperfinite.

The second part of the paper deals with Toeplitz subshifts of residually finite, non-amenable groups. For those, we prove that topological conjugacy is somewhat complicated.

Theorem 1.2. *If G is a non-amenable, residually finite group, then the equivalence relation of conjugacy on the set of free, Toeplitz G-subshifts is not hyperfinite.*

The proof of this theorem proceeds by constructing a probability measure μ on the set of Toeplitz subshifts which is invariant under a suitable action of the group G, included in the equivalence relation; then we show that the stabilizers of points are amenable and conclude that the equivalence relation is not μ-amenable and thus not hyperfinite.

In the last section of the paper, reinterpreting a result of Downarowicz, Kwiatkowski, and Lacroix, we indicate how topological conjugacy of Toeplitz $\mathbb{Z}$-subshifts is naturally generated by an action of a groupoid which is "hyperfinite-by-compact." This seems to indicate that the equivalence relation is somewhat simple; however, we are not even able to prove that it is not universal. The following question remains open:

Question 1.3. Is topological conjugacy of Toeplitz $\mathbb{Z}$-subshifts hyperfinite? Is this true for an arbitrary residually finite, amenable group?

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2. Toeplitz subshifts

In this section, we recall the definition and collect some basic properties of Toeplitz subshifts for residually finite groups. We also establish some basic definability properties that will be needed later.

Let G be a countable group. Recall that the *profinite topology* on G is the one with the basis of neighborhoods at 1_G consisting of all finite index subgroups (or, equivalently, all finite index *normal* subgroups). G is called *residually finite* if $\{1_G\}$ is closed in the profinite topology, or, equivalently, if the profinite topology is Hausdorff. For the rest of the paper, G will always be a residually finite group.

The *profinite completion* of G, denoted by $\widehat{G}$, is the completion of the group uniformity defined by this topology; equivalently $\widehat{G} = \varprojlim G/H$, where the limit is taken over all finite index normal subgroups of G. This profinite completion is metrizable if G has only countably many subgroups of finite index (for example, if it is finitely generated); even though it is not strictly necessary for what we are doing (as we can always pass to suitable metrizable quotients), we will sometimes assume this for convenience.

Recall that the *left shift action* $G \curvearrowright 2^G$ is defined by $(g \cdot x)(h) = x(g^{-1}h)$. A closed subset $S \subseteq 2^G$ is called a *subshift* if it is invariant under this action.

Definition 2.1 (Krieger [17])**.** A word $x \in 2^G$ is called *Toeplitz* if $x \colon G \to 2$ is a continuous map for the profinite topology on G (where $2 = \{0, 1\}$ is taken to be discrete). A subshift $S \subseteq 2^G$ is *Toeplitz* if there exists a Toeplitz word x such that $S = \overline{G \cdot x}$.

Generalizing a well-known fact for $G = \mathbb{Z}$, Krieger showed that every Toeplitz subshift is minimal (see [17, Corollaire 2.5]).

Let

$$S(G) = \{S \subseteq 2^G : S \text{ is closed and } G\text{-invariant}\}$$

be the set of all G-subshifts. This is naturally a compact space with the Vietoris topology. It is easy to check that the family of minimal subshifts as well as that of free subshifts form Borel subsets of $S(G)$. In what follows, we see that being Toeplitz is also a Borel condition and establish a simple selection lemma that will be used later.

We need the following well-known definability property of Baire category notions for which we have not been able to find a suitable reference. It is a slight generalization of [15, 16.1] with the same proof. If X is a Polish space, $F(X)$ denotes the Effros Borel space of closed subsets of X. Recall that $\exists^* x \in F$ means "for non-meagerly many x in F."

Proposition 2.2. (i) *Let $(X, \mathcal{S})$ be a measurable space, Y a Polish space and let $\Phi \colon X \to F(Y)$ be a measurable function, where $F(Y)$ denotes the Effros Borel space of closed subsets of Y. Let $A \subseteq X \times Y$ be a measurable set (where Y is equipped with its Borel σ-algebra) and $U \subseteq Y$ an open set. Then the set*

$$\{x \in X : U \cap \Phi(x) = \emptyset \text{ or } A_x \cap \Phi(x) \text{ is non-meager in } U \cap \Phi(x)\} \tag{1}$$

is measurable.

(ii) *Let X be a Polish space and $A \subseteq X$ be Borel. Then the set*

$$\{F \in F(X) : \exists^* x \in F \ x \in A\}$$

is Borel.

Proof. (i). The proof goes exactly as in [15, 16.1]. If we let A_U denote the set defined in (1), one checks that

- for all $S \in \mathcal{S}$, $V \subseteq Y$ open
$$(S \times V)_U = \{x \in X : U \cap \Phi(x) = \emptyset \text{ or } (x \in S \text{ and } U \cap V \cap \Phi(x) \neq \emptyset)\};$$
- $(\bigcup_n A_n)_U = \bigcup_n (A_n)_U$;
- $(\sim A)_U = \sim \bigcap_{U_n \subseteq U} A_{U_n}$, where the intersection is over all $U_n \subseteq U$ from a fixed countable base of Y.

(ii). This follows from (i). □

If x is a Toeplitz word and $H \leq G$ is a finite index subgroup, we let

$$\mathrm{Per}_H(x) = \{g \in G : x|_{Hg} \text{ is constant}\}.$$

The fact that x is Toeplitz translates to $\bigcup_H \mathrm{Per}_H(x) = G$. H is called an *essential group of periods* for x if for all $g \notin H$, $\mathrm{Per}_H(x) \nsubseteq \mathrm{Per}_H(g \cdot x)$.

The *maximal equicontinuous factor* (*m.e.f.* for short) of a topological dynamical system $G \curvearrowright X$ is the factor generated by all equicontinuous factors of X. For a Toeplitz subshift S, this is always of the form $G \curvearrowright \varprojlim G/H_n$, where $\{H_n\}$ is a

decreasing sequence of essential groups of periods (see [3, Proposition 7]). Note that every such system can also be written as $G \curvearrowright \widehat{G}/K$, where $K = \bigcap_n \overline{H_n}$ with the closures taken in $\widehat{G}$. We have the following folklore lemma.

Lemma 2.3. *Let S be a Toeplitz subshift. Then the set of Toeplitz words in S is dense G_δ.*

Proof. Let Y be the m.e.f. of S and $\pi \colon S \to Y$ be the factor map. By [3, Theorem 2], the set of Toeplitz words in S can be written as

$$\{x \in S : \pi^{-1}(\{\pi(x)\}) = \{x\}\}.$$

This can be written as $\pi^{-1}(A)$, where

$$A = \bigcap_{\epsilon > 0} \bigcup \{U \subseteq Y \text{ open} : \operatorname{diam}(\pi^{-1}(U)) < \epsilon\}$$

and this is clearly G_δ. It is also dense by the definition of a Toeplitz subshift. □

Denote

$$\text{Töp}(G) = \{S \in S(G) : S \text{ is a Toeplitz subshift}\}.$$

Lemma 2.3 allows us to build the following selector map that will be used throughout the paper.

Proposition 2.4. *Let G be a residually finite group. Then there exists a Borel map $\tau \colon \text{Töp}(G) \to 2^G$ such that for all $S \in \text{Töp}(G)$, $\tau(S) \in S$ and $\tau(S)$ is a Toeplitz word.*

Proof. This follows from Lemma 2.3, Proposition 2.2, and [15, 18.6]. □

Now we can easily deduce the following.

Lemma 2.5. *Let G be a residually finite group such that $\widehat{G}$ is metrizable. The map $\text{Töp}(G) \to F(\widehat{G})$ which associates to a Toeplitz subshift S (the conjugacy class of) the subgroup $K \leq \widehat{G}$ such that the m.e.f. of S is isomorphic to $\widehat{G}/K$ is Borel.*

Proof. By Proposition 2.4, for a given Toeplitz subshift S, we can choose in a Borel way a Toeplitz word $x \in S$. By [3], we can choose in a Borel way a sequence $L_n(x)$ of finite index subgroups of G, which are essential periods of x and such that, posing $K = \bigcap_n \overline{L_n}$, we have that the m.e.f. of S is isomorphic to $\widehat{G}/K$. Now it is easy to check that the map that associates to the sequence $\{L_n\}_n$ the intersection $\bigcap_n \overline{L_n}$ is Borel. □

Proposition 2.6. *Let G be a residually finite group, $\widehat{G}$ the profinite completion of G and $K \leq \widehat{G}$ a closed subgroup. Then the set $\text{Töp}(G)$ is Borel and if $\widehat{G}$ is metrizable,*

$$\text{Töp}(G, K) = \{S \in K(2^G) : S \textit{ is a Toeplitz subshift with m.e.f. } \widehat{G}/K\}$$

is also Borel.

Proof. We have

$$S \in \text{Töp}(G) \iff \exists^* x \in S \ x \text{ is Toeplitz}$$

and applying Proposition 2.2 yields that $\text{Töp}(G)$ is a Borel set.

The second statement follows from Lemma 2.5. □

3. $\mathbb{Z}$-subshifts

In this section, we consider $\mathbb{Z}$-subshifts and prove Theorem 1.1.

Let $x \in 2^{\mathbb{Z}}$ be a Toeplitz word and $p \in \mathbb{N}$. Denote by $\mathrm{Per}_p(x)$ the subset of $\mathbb{Z}/p\mathbb{Z}$ defined as follows:

$$\mathrm{Per}_p(x) = \{i + p\mathbb{Z} : x(i + kp) = x(i) \text{ for all } k \in \mathbb{Z}\}.$$

Let $H_p(x) = (\mathbb{Z}/p\mathbb{Z}) \setminus \mathrm{Per}_p(x)$. The elements of $H_p(x)$ are called *p-holes* for x.

We say that x *has separated holes* [8] if

$$\lim_{p\to\infty} \min\{|i - j| : i, j \in H_p(x), i \neq j\} = \infty.$$

Having separated holes is a property of the subshift that does not depend on the choice of the word x. Thus, by the arguments in Section 2, the condition of having separated holes defines a Borel subset of the set of all subshifts. All shifts with separated holes are regular and have topological entropy 0.

Next we recall the definition of an *amenable* equivalence relation. Let E be a countable equivalence relation on the standard Borel space X. If $x \in X$ and $f\colon E \to \mathbb{R}$ is a function, denote by f_x the function $[x]_E \to \mathbb{R}$ defined by $f_x(y) = f(x, y)$. E is called *(1-)amenable* if there exist positive Borel functions $\lambda^n\colon E \to \mathbb{R}$ such that

- $\lambda^n_x \in \ell^1([x]_E)$, $\|\lambda^n_x\|_1 = 1$;
- $\lim_{n\to\infty} \|\lambda^n_x - \lambda^n_y\|_1 = 0$ for all $(x, y) \in E$.

By a theorem of Connes, Feldman and Weiss [2] (see also [16]), if μ is any probability measure on X and E is amenable, then it is hyperfinite μ-almost everywhere. It is an open question whether every amenable equivalence relation is hyperfinite. See [13] for more details on amenable equivalence relations in the Borel setting.

The goal of this section is to prove the following theorem.

Theorem 3.1. *The equivalence relation of isomorphism of Toeplitz $\mathbb{Z}$-subshifts with separated holes is amenable.*

Proof. Let $p \in \mathbb{N}$. Denote by $\mathrm{Sym}(2^p)$ the set of all bijections $2^p \to 2^p$. For $\pi \in \mathrm{Sym}(2^p)$, let $\hat{\pi}\colon 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be defined by $\hat{\pi}(x)|_{[kp,(k+1)p)]} = \pi(x|_{[kp,(k+1)p)})$ for all $k \in \mathbb{Z}$. Then if $S \subseteq 2^{\mathbb{Z}}$ is a closed $p\mathbb{Z}$-invariant set, $\hat{\pi}(S)$ is also a closed $p\mathbb{Z}$-invariant set and $\hat{\pi}$ defines an isomorphism between them (as $p\mathbb{Z}$-systems).

If $x \in 2^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, denote by $x + i$ the shift of x by i. If $S \subseteq 2^{\mathbb{Z}}$ is a Toeplitz subshift, $x \in S$ is a Toeplitz word and p is a period, define $A_{p,i}(x) \subseteq S(\mathbb{Z})$ and $B^p\colon E \to \mathbb{R}$ by

$$A_{p,i}(x) = \{\overline{\mathbb{Z} \cdot \hat{\pi}(x + i)} : \pi \in \mathrm{Sym}(2^p)\},$$
$$B^p(S, T) = \sum_{i<p} \chi_{A_{p,i}(x)}(T),$$

where χ_A denotes the characteristic function of A. Note that the value of $B^p(S, T)$ does not depend on the choice of a Toeplitz $x \in S$. Indeed, let $x_1, x_2 \in S$ be Toeplitz. We have that $S = \bigcup_{j<p} \overline{p\mathbb{Z} \cdot (x_1 + j)}$ and there exists j_0 such that $x_2 \in \overline{p\mathbb{Z} \cdot (x_1 + j_0)}$ and therefore (as $\overline{p\mathbb{Z} \cdot (x_1 + j_0)}$ is Toeplitz and thus, minimal), $\overline{p\mathbb{Z} \cdot x_2} = \overline{p\mathbb{Z} \cdot (x_1 + j_0)}$. Then $\overline{p\mathbb{Z} \cdot \hat{\pi}(x_2 + i)} = \overline{p\mathbb{Z} \cdot \hat{\pi}(x_1 + j_0 + i)}$, whence, by minimality, $\overline{\mathbb{Z} \cdot \hat{\pi}(x_2 + i)} = \overline{\mathbb{Z} \cdot \hat{\pi}(x_1 + j_0 + i)}$ for every i and every π. Note finally that, as we can choose $x = \tau(S)$ as per Proposition 2.4, the function B^p is Borel.

Denote by E the equivalence relation of isomorphism on Toeplitz subshifts and define $\lambda^p\colon E \to \mathbb{R}$ by

$$\lambda^p(S, T) = \frac{B^p(S, T)}{\sum_{T' \, E \, S} B^p(S, T')} = \frac{B^p(S, T)}{\|B^p_S\|_1}.$$

The functions λ^p clearly satisfy the first condition in the definition of amenability; in what follows, we check that if we restrict to the subshifts with separated holes, they also satisfy the second.

Recall that if S and T are subshifts and $f\colon S \to T$ is an isomorphism, then it is given by a *block code*, i.e., there exists $r \in \mathbb{N}$ and a function $\phi\colon 2^{2r+1} \to 2$ such that for all $x \in S$,

$$f(x)(i) = \phi(x|_{[i-r,i+r]}).$$

The number r is called the *radius* of the block code.

Let S and T be two Toeplitz subshifts with separated holes and let $f\colon S \to T$ be an isomorphism such that f and f^{-1} are given by block codes of radius r. Let $\epsilon > 0$ be given. If $C \subseteq \mathbb{Z}$, let

$$(C)_r = \{c + j \in \mathbb{Z}/p\mathbb{Z} : c \in C, |j| \leq r\}.$$

(Here and below, we suppress the natural quotient map $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ from the notation.) Note that $|(C)_r| \leq (2r+1)|C|$.

Let $x \in S$ be a Toeplitz word and let $y = f(x)$. Let p be a period so big that the distance between two consecutive holes in $H_p(x)$ and in $H_p(y)$ is larger than $M = (2r+1)/\epsilon$.

Claim 1. *For all $i \notin (H_p(x))_r$, we have $A_{p,i}(x) = A_{p,i}(y)$.*

Proof. This follows from the fact that for $i \notin (H_p(x))_r$, there exists $\pi \in \mathrm{Sym}(2^p)$ such that $\hat{\pi}(x+i) = y+i$. Indeed, as $y = f(x)$ and f is given by a block code of radius r, $y|_{[i+kp,i+(k+1)p)}$ only depends on $x|_{[i+kp-r,i+(k+1)p+r)}$ and

$$\begin{aligned} x|_{[i+kp-r,i+kp)]} &= x|_{[i+(k+1)p-r,i+(k+1)p)} \\ x|_{[i+(k+1)p,i+(k+1)p+r)} &= x|_{[i+kp,i+kp+r)} \end{aligned}$$

because $(i-r, i+r) \subseteq \mathrm{Per}_p(x)$. Thus $y|_{[i+kp,i+(k+1)p)}$ can be calculated from $x|_{[i+kp,i+(k+1)p)}$ in a way independent of k which gives the claim. □

Claim 2. *For all $i \notin H_p(x)$, we have $A_{p,i}(x) = A_{p,i+1}(x)$.*

Proof. Indeed, let $\sigma\colon 2^{\mathbb{Z}/p\mathbb{Z}} \to 2^{\mathbb{Z}/p\mathbb{Z}}$ be defined by $\sigma(z)(j) = z(j-1)$, and observe that as $i \in \mathrm{Per}_p(x)$, $\hat{\sigma}(x+i) = x+i+1$. Then

$$\begin{aligned} A_{p,i}(x) &= \{\overline{\mathbb{Z} \cdot \hat{\pi}(x+i)} : \pi \in \mathrm{Sym}(2^p)\} \\ &= \{\overline{\mathbb{Z} \cdot (\widehat{\pi\sigma})(x+i)} : \pi \in \mathrm{Sym}(2^p)\} \\ &= A_{p,i+1}(x). \end{aligned}$$ □

Let $h_0, h_1, h_2, \dots h_J$ enumerate the holes in $H_p(x)$ in circular order (so that $J+1 = 0$). By the choice of p, $|h_j - h_{j+1}| \geq M$ for all j. If f is a real-valued function, denote by f^+ the function $\max(f, 0)$ and note that $\|f\|_1 = \|f^+\|_1 + \|(-f)^+\|_1$. We

calculate:

$$\begin{aligned}
\|(B^p_S - B^p_T)^+\|_1 &= \Big\|\Big(\sum_{i<p}\chi_{A_{p,i}(x)} - \sum_{i<p}\chi_{A_{p,i}(y)}\Big)^+\Big\|_1 \\
&\le \Big\|\sum_{j<J}\sum_{i=h_j-r}^{h_j+r}\chi_{A_{p,i}(x)}\Big\|_1 \\
&\le \Big\|\sum_{j<J}\Big(\sum_{i=h_j+1}^{h_j+r}\chi_{A_{p,i}(x)} + \sum_{i=h_{j+1}-r}^{h_{j+1}}\chi_{A_{p,i}(x)}\Big)\Big\|_1 \\
&\le \sum_{j<J}\big((2r+1)/M\big)\sum_{i=h_j+1}^{h_{j+1}}|A_{p,i}(x)| \\
&= \big((2r+1)/M\big)\|B^p_S\|_1 \\
&\le \epsilon\|B^p_S\|_1.
\end{aligned}$$

For the inequality on the second line, we use Claim 1, and for the one on the fourth line, we use Claim 2. Similarly, $\|(B^p_T - B^p_S)^+\|_1 \le \epsilon\|B^p_T\|_1$, whence

$$\|B^p_S - B^p_T\|_1 \le \epsilon\big(\|B^p_S\|_1 + \|B^p_T\|_1\big).$$

Using these estimates, it is now easy to see that if $\{p_n\}$ is a sequence of periods such that $p_n|p_{n+1}$ for all n, then $\|\lambda^{p_n}_x - \lambda^{p_n}_y\|_1 \to 0$ as $n \to \infty$. □

4. The non-amenable case

Now we return to the general situation of Section 2, where G is a residually finite group. We also fix a decreasing sequence $\{H_n\}$ of finite index, *normal* subgroups with trivial intersection, to be determined later. We also denote by $\hat{G}$ the inverse limit $\varprojlim G/H_n$ (which is a group quotient of the profinite completion $\widehat{G}$) and by $\pi_n : \hat{G} \to G/H_n$ the natural projections.

The goal of this section is to prove Theorem 1.2. The proof proceeds by constructing a probability measure on the set of Toeplitz subshifts which is invariant under an appropriate action of G contained in the isomorphism relation. We then check that the point stabilizers for this action are amenable and conclude that if G is non-amenable, then the isomorphism equivalence relation is not amenable either.

4.1. **The left and right actions.** First, we describe the construction of the measure. Let $A_n = (H_{n-1}/H_n) \setminus \{H_n\}$ and let $Z = 2^{\bigsqcup_n A_n}$. Let $Y = \{y \in \hat{G} : y \notin G\}$. Consider the maps

$$Y \times Z \xrightarrow{\sigma} 2^G \xrightarrow{\rho} S(G)$$

defined as follows. If $(y, z) \in Y \times Z$, define $\sigma(y, z)$ by

$$\sigma(y, z)(h) = z(\pi_{n_0}(y^{-1}h)), \quad \text{where } n_0 = \min\{n : \pi_n(y) \neq \pi_n(h)\}.$$

Define $\rho: 2^G \to S(G)$ by $\rho(x) = \overline{G \cdot x}$. Let $\theta = \rho \circ \sigma$.

G acts on 2^G on both sides: on the left, by

$$(g \cdot x)(h) = x(g^{-1}h)$$

and on the right, by

$$(x \cdot g)(h) = x(hg^{-1})$$

and the two actions commute. Note that if $S \subseteq 2^G$ is a subshift and $g \in G$, then $S \cdot g$ is a subshift too, so we have a right action $S(G) \curvearrowleft G$. Moreover, for every fixed $g \in G$, the map

$$S \to S \cdot g, \quad x \mapsto x \cdot g$$

is an isomorphism of subshifts.

The space $Y \times Z$ is also equipped with commuting left and right actions of G, defined as follows. For $y \in Y$ and $g \in G$, define $g \cdot y = gy$ and $y \cdot g = yg$ (recall that $G \subseteq \hat{G}$). For $z \in Z$, define $g \cdot z = z$ and $(z \cdot g)(a) = z(gag^{-1})$ for all $a \in \bigsqcup_n A_n$. Equip $Y \times Z$ with the diagonal left and right actions.

Lemma 4.1. *The maps σ and ρ commute with both the left and right actions of G on the respective spaces.*

Proof. We only check that σ commutes with the right actions. Suppose that $(y_1, z_1), (y_2, z_2)$ are elements of $Y \times Z$ such that $(y_1, z_1) \cdot g = (y_2, z_2)$. Let $x_1 = \sigma(y_1, z_1)$ and $x_2 = \sigma(y_2, z_2)$. Note that for every $h \in G$ we have

$$\min\{n : \pi_n(y_1) \neq \pi_n(hg^{-1})\} = \min\{n : \pi_n(y_1 g) \neq \pi_n(h)\}.$$

Write $n_0(h)$ for this common value. Thus,

$$\begin{aligned} (x_1 \cdot g)(h) = x_1(hg^{-1}) &= z_1(\pi_{n_0(h)}(y_1^{-1}hg^{-1})), \quad \text{and} \\ x_2(h) = z_2(\pi_{n_0(h)}(y_2^{-1}h)) &= z_2(\pi_{n_0(h)}(g^{-1}y_1^{-1}h)). \end{aligned}$$

But

$$\begin{aligned} z_2(\pi_{n_0(h)}(g^{-1}y_1^{-1}h)) &= z_1(g\pi_{n_0(h)}(g^{-1}y_1^{-1}h)g^{-1})) \\ &= z_1(\pi_{n_0(h)}(y_1^{-1}hg^{-1})), \end{aligned}$$

so $x_1 \cdot g = x_2$, as needed. □

We call an element $z \in Z$ *proper* if it takes both values 0 and 1 infinitely many times. Note that z is proper if and only if for every (any) $y \in Y$, $G \cdot \sigma(y, z)$ is infinite.

Lemma 4.2. *For any $y \in Y$ and proper $z \in Z$, $\sigma(y, z)$ is a Toeplitz word and the m.e.f. of $\overline{G \cdot \sigma(y, z)}$ is isomorphic to $\hat{G}$. In particular, the subshift $\overline{G \cdot \sigma(y, z)}$ is free.*

Proof. Write $x = \sigma(y, z)$. First note that for any $g \in G$, the value $x(g)$ is assumed on the whole H_{n_0}-coset $\pi_{n_0}(y^{-1}g)$, where $n_0 = \min\{n : \pi_n(y) \neq \pi_n(g)\}$, showing that x is Toeplitz.

To calculate the m.e.f., by [3], it suffices to observe that for all n, H_n is an essential group of periods for x. Indeed, using the fact that z is proper, we have that $\mathrm{Per}_{H_n}(x) = G \setminus \pi_n(y)$ which is not contained in $\mathrm{Per}_{H_n}(g \cdot x) = G \setminus \pi_n(g \cdot y)$ for any $g \notin H_n$.

For the final claim, just note that as $\bigcap_n H_n = \{1_G\}$ and the H_n are normal, the translation action $G \curvearrowright \hat{G}$ is free. □

The main ingredient for the proof of the theorem is the following lemma.

Lemma 4.3. *For all proper $z_1, z_2 \in Z$ and all $y_1, y_2 \in Y$,*

$$z_1 \neq z_2 \implies \theta(y_1, z_1) \neq \theta(y_2, z_2).$$

Proof. Let $x_i = \sigma(y_i, z_i)$. To show that $\overline{G \cdot x_1} \neq \overline{G \cdot x_2}$, it suffices to find a "sub-word" of x_2 that does not occur in x_1, i.e., a finite set $F \subseteq G$ such that

$$\forall g \in G\ \exists f \in F \quad x_1(gf) \neq x_2(f).$$

Let $a_0 \in A_n$ be such that $z_1(a_0) \neq z_2(a_0)$. By replacing (y_1, z_1) and (y_2, z_2) with $g_1 \cdot (y_1, z_1)$ and $g_2 \cdot (y_2, z_2)$ (which does not change either z_i or $\theta(y_i, z_i)$) for suitably chosen $g_1, g_2 \in G$, we may assume that $\pi_n(y_1) = \pi_n(y_2) = 1_{G/H_n}$. Let $f_0, f_1 \in G$ be such that $\pi_n(f_0) = \pi_n(f_1)$, $x_2(f_0) = 0$, $x_2(f_1) = 1$ (such f_0, f_1 exist by the assumption that z_2 is proper). Let $f \in G$ be such that $\pi_n(f) = a_0$ and finally, let $F = \{f, f_0, f_1\}$.

Next we show that this F works. Let $g \in G$ be arbitrary. We distinguish two cases: $g \in H_n$ and $g \notin H_n$. Let first $g \in H_n$. We check that $x_1(gf) \neq x_2(f)$. Indeed,

$$x_1(gf) = z_1(\pi_n(y_1^{-1}gf)) = z_1(\pi_n(f)) = z_1(a_0)$$

and

$$x_2(f) = z_2(\pi_n(y_2^{-1}f)) = z_2(\pi_n(f)) = z_2(a_0),$$

which are different by the choice of a_0.

Suppose now that $g \notin H_n$. We will show that $x_1(gf_0) = x_1(gf_1)$ which will complete the proof (as $x_2(f_0) \neq x_2(f_1)$). Indeed, observe that the least k for which $\pi_k(gf_i) \neq 1_{G/H_k}$ is at most n and is the same for $i = 0, 1$ (as $\pi_n(f_0) = \pi_n(f_1)$). Recall also that $\pi_k(y_1) = \pi_k(y_2) = 1_{G/H_k}$. Now we have

$$\begin{aligned} x_1(gf_0) &= z_1(\pi_k(y_1^{-1}gf_0)) = z_1(\pi_k(gf_0)) \\ &= z_1(\pi_k(gf_1)) = x_1(gf_1), \end{aligned}$$

completing the proof. ☐

4.2. **A.e. amenable stabilizers.** Let λ be the Haar measure on $\hat{G}$ and note that as $\lambda(Y) = 1$, we can consider λ as a measure on Y. Let ν be the Bernoulli $(\frac{1}{2}, \frac{1}{2})$ measure on $Z = 2^{\bigsqcup_n A_n}$. Equip $Y \times Z$ with the product measure $\lambda \times \nu$. It is clear that this measure is invariant under both the left and the right action of G (but for us it will be the right one that will be important). Let $\mu = \theta_*(\lambda \times \nu)$.

Lemma 4.4. *The measure μ concentrates on the set of free, Toeplitz G-subshifts and is invariant under the right action.*

Proof. This follows from Lemma 4.2 and the fact that ν concentrates on the set of proper elements of Z. Invariance follows directly from Lemma 4.1. ☐

For $g \in G$, denote by $C(g)$ the *centralizer* of g in G, i.e., the set of all elements of G that commute with g. Let

$$Z_f(G) = \{g \in G : C(g) \text{ has finite index in } G\}.$$

Lemma 4.5. *$Z_f(G)$ is an amenable, normal subgroup of G.*

Proof. It is clear that $Z_f(G)$ is a normal subgroup of G. To see that it is amenable, note that the centralizers of all elements in $Z_f(G)$ have finite index in $Z_f(G)$. Hence, $Z_f(G)$ has finite conjugacy classes and this implies amenability (Leptin [18]). ☐

Next we choose a suitable sequence $\{H_n\}$ of finite index, normal subgroups of G that will allow us to prove the theorem. Note that the residual finiteness of G implies that for any finite index subgroup $H \lhd G$ and $g \in G \setminus Z_f(G)$ there exists a finite index subgroup $H' \leq H$ normal in G such that for some $C \in H/H'$ we have $gCg^{-1} \neq C$. Enumerate $G \setminus Z_f(G)$ as $\{g_n : n \in \mathbb{N}\}$ and construct inductively $\{H_n : n \in \mathbb{N}\}$ a decreasing sequence with trivial intersection such that

$$\text{for all } m \geq n \text{ there exists } C \in H_m/H_{m+1} \text{ such that } g_n^{-1}Cg_n \neq C. \tag{2}$$

Proof of Theorem 1.2. We will show that the right action of G on $\text{Töp}(G)$ has μ-a.e amenable stabilizers. Recall that (see, e.g., [11, Lemma 3.6]) if a measure-preserving action of a group has amenable stabilizers and induces a hyperfinite equivalence relation, then the group is amenable. Thus, if G is non-amenable, we obtain that the orbit equivalence relation induced by the right action of G on the set of Toeplitz G-subshifts is not hyperfinite and, as it is contained in the topological conjugacy relation, the latter is not hyperfinite either.

By Lemma 4.5, it is enough to show that there is a measure 1 subset A of $\mathrm{Töp}(G)$ such that the stabilizer of every element in A is contained in $Z_f(G)$. Since G is countable, it suffices to see that for every $g \in G \setminus Z_f(G)$, the set

$$\{(y,z) \in Y \times Z : \theta(y,z) \cdot g \neq \theta(y,z)\}$$

has measure 1. By Lemma 4.3, it is enough to show that $\{z \in Z : z \cdot g \neq z\}$ is of measure 1. As $g \notin Z_f(G)$, by (2), for almost all n, there is an element, say $C_n \in H_{n+1}/H_n$ such that $gC_ng^{-1} \neq C_n$. By the definition of the action of G on Z, for any $z \in Z$ such that there exists n with $z(C_n) \neq z(gC_ng^{-1})$, we have $z \neq z \cdot g$. But the latter condition is clearly satisfied on a measure 1 set. This completes the proof of the theorem. □

Remark 4.6. Note that in case G is a non-cyclic free group, the right action of G on $\mathrm{Töp}(G)$ is free μ-a.e. because $Z_f(G)$ is trivial (centralizers of non-trivial elements of free groups are cyclic). This implies in particular that the equivalence relation of topological conjugacy of Toeplitz G-subshifts embeds a free, measure-preserving action of a free group.

5. The groupoid viewpoint

The equivalence relation of isomorphism of subshifts is naturally given by an action of a *groupoid* rather than a group. Recall that a *groupoid* is a small category where each arrow is invertible. We will only be interested in countable Borel groupoids, defined as follows (see also [20]). A *countable Borel groupoid* Γ is a tuple $(X, A, s, r, \circ)$, where X is a standard Borel space of objects, A is a standard Borel space of arrows, $s\colon A \to X$ is a Borel map specifying the *source* of each arrow and $r\colon A \to X$ a Borel map specifying its *range*. $\circ$ is a partial Borel map $A \times A \to A$ which represents the composition of arrows; $f \circ g$ is defined whenever $r(g) = s(f)$. Of course, we require that $(X, A, \circ)$ be a groupoid. Furthermore, we require that $r^{-1}(\{x\})$ and $s^{-1}(\{x\})$ be countable sets for every $x \in X$. By the Lusin–Novikov selection theorem, there exist sequences of Borel maps $P_n\colon X \to A$ and $Q_n\colon X \to A$ such that $P_n(x)$ enumerate $s^{-1}(x)$ and $Q_n(x)$ enumerate $r^{-1}(x)$ for every x. If $x, y \in X$, we will denote by $A(x,y)$ the set of arrows between x and y. We will sometimes identify a groupoid with the set of its arrows as the other information can be recovered from it.

Every countable Borel groupoid Γ gives rise to a countable Borel equivalence relation E_Γ on X defined by

$$x \mathrel{E_\Gamma} y \iff \exists f \in A \ s(f) = x \text{ and } r(f) = y.$$

Conversely, every countable equivalence relation can be viewed as groupoid, where there is a unique arrow between every pair of equivalent elements of X.

Now let G be a residually finite group as before, $\{H_n\}$ be a sequence of finite index, normal subgroups of G, and let X be the standard Borel space of of all Toeplitz subshifts of G with m.e.f. $\hat{G} = \varprojlim G/H_n$. Let A be the set of all isomorphisms between elements of X. Recalling that every such isomorphism is given by a block code (which is a finite object), it is easy to endow A with a standard Borel structure so that $(X, A, \circ)$ becomes a countable Borel groupoid ($\circ$ is just composition of maps). The equivalence relation generated by this groupoid is exactly isomorphism of subshifts. (All of this is defined for arbitrary subshifts of countable groups; however, below we will need that they be Toeplitz.)

Note that for every $S \in X$ and every $x \in S$, there exists a unique factor map $\pi_x\colon S \to \hat{G}$ such that $\pi(x) = 1_{\hat{G}}$. (Existence follows from the fact that if $\pi\colon S \to \hat{G}$ is any G-map, then post-composing π with right multiplication by $\pi(x)^{-1}$ yields a map that sends x to $1_{\hat{G}}$.) Let $\tau\colon X \to 2^G$ be a Borel map that selects for every

$S \in X$, a Toeplitz word $\tau(S) \in S$ as given by Proposition 2.4. Now we can define a Borel cocycle $\alpha_0 \colon \Gamma \to \hat{G}$ by

$$\alpha_0(f) = \pi_{\tau(r(f))}(f(\tau(s(f))))^{-1}.$$

Recall that a *cocycle* is just a map $\Gamma \to \hat{G}$ that satisfies $\alpha(f \circ g) = \alpha(f)\alpha(g)$. Two cocycles α, β are *cohomologous* if there exists a Borel map $F \colon X \to \hat{G}$ such that $\beta(f) = F(r(f))^{-1}\alpha(f)F(s(f))$ for all $f \in \Gamma$. Note that changing the selection map τ transforms α_0 into a cohomologous cocycle. We also have a natural Borel homomorphism $\rho \colon \Gamma \to E$ defined by

$$\rho(f) = (s(f), r(f)).$$

Now we specialize to the case $G = \mathbb{Z}$. We have the following proposition, which basically follows from a result by Downarowicz, Kwiatkowski, and Lacroix [6].

Proposition 5.1. *Let $\Delta = \ker \alpha_0 = \{f \in \Gamma : \alpha_0(f) = 0\}$. Then the equivalence relation E_Δ is hyperfinite.*

Proof. For a period p, define the finite equivalence relation E_p on $\mathrm{Töp}(\mathbb{Z})$ by

$$S \, E_p \, T \iff \exists \sigma \in \mathrm{Sym}(2^p) \ \hat{\sigma}(\tau(S)) = \tau(T).$$

It follows from [6, Theorem 1] that

$$\begin{aligned} E_\Delta &= \{(S, T) : \exists f \colon S \to T \text{ isomorphism such that } f(\tau(S)) = \tau(T)\} \\ &= \bigcup_p E_p, \end{aligned}$$

showing that E_Δ is hyperfinite. ☐

If S is a subshift, the *centralizer* $C(S)$ of S is the group of automorphisms of S, or equivalently, the group of arrows from S to itself. Note that by our observations above, if $S \in X$, then $C(S)$ embeds in $\hat{G}$ and is therefore an abelian group. So, in some sense, the grooupoid Γ differs from the equivalence relation E_Γ only a little.

We finally observe that the existence of the cocycle α_0 gives some restrictions on the groupoid Γ. For example, using Popa's cocycle superrigidity results [22], it is easy to prove that Γ does not embed the groupoid of the free part of any Bernoulli action of an infinite property (T) group. However, it is not clear how to conclude anything from that about the equivalence relation E; in particular, we do not know whether E is universal.

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